

SUBSTITUTIONS, TILING DYNAMICAL SYSTEMS AND MINIMAL SELF-JOININGS

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ABSTRACT. We investigate substitution subshifts and tiling dynamical systems arising from the substitutions (1) $\theta : 0 \rightarrow 001, 1 \rightarrow 11001$ and (2) $\eta : 0 \rightarrow 001, 1 \rightarrow 11100$. We show that the substitution subshifts arising from θ and η have minimal self-joinings and are mildly mixing. We also give a criterion for 1-dimensional tiling systems arising from θ or η to have minimal self-joinings. We apply this to obtain examples of mildly mixing 1-dimensional tiling systems.

1. INTRODUCTION

A measure preserving system, denoted by $\mathbf{X} = (X, \mathcal{X}, \mu, T_G)$, consists of a probability space (X, \mathcal{X}, μ) together with a measure preserving action $T_G = (T_g)_{g \in G}$ of a locally compact group G . In this paper we mainly consider the cases $G = \mathbb{Z}$ and $G = \mathbb{R}$, and denote the corresponding measure preserving system by $\mathbf{X} = (X, \mathcal{X}, \mu, T)$ or $\mathbf{X} = (X, \mathcal{X}, \mu, (T_t)_{t \in \mathbb{R}})$ respectively. For $G = \mathbb{R}$, we also assume that G acts continuously in the sense that for every $A \in \mathcal{X}$ and $\epsilon > 0$, there exists a neighborhood V of the identity in G such that $\mu(A \Delta T_g A) < \epsilon$ for $g \in V$. A measure preserving system \mathbf{X} is called

- (1) *ergodic* if any set A with $\mu(A \Delta T_g A) = 0$ for all $g \in G$ is of measure 0 or 1.
- (2) *weakly mixing* if the diagonal action $(T_g \times T_g)_{g \in G}$ on $(X \times X, \mathcal{X} \otimes \mathcal{X}, \mu \otimes \mu)$ is ergodic. For an abelian group G , it is equivalent to the fact that the only eigenfunctions of T_G are constant functions. Recall that $f \in L^2(X)$ is called an eigenfunction if $f(T_g x) = \lambda(g)f(x)$ for some $\lambda \in \hat{G}$, the dual group of G .
- (3) *mildly mixing* if there is no non-constant rigid function. Recall that a function $f \in L^2(X)$ is called *rigid* if there is a sequence $t_i \in G$ with $t_i \rightarrow \infty$ such that $f \circ T_{t_i}$ converges to f in $L^2(X)$. (We write $t_i \rightarrow \infty$ if, for any compact set $K \subset G$, there is a positive integer N such that $t_i \notin K$ if $i \geq N$.)
- (4) *rigid* if there is a sequence $t_i \in G$ with $t_i \rightarrow \infty$ such that for every $f \in L^2(X)$, $f \circ T_{t_i}$ converges to f in $L^2(X)$.
- (5) *strongly mixing* if for any $f_1, f_2 \in L^2(X)$ and for any sequence (t_i) with $t_i \rightarrow \infty$,

$$\lim_{t_i \rightarrow \infty} \int f_1(x) f_2(T_{t_i} x) d\mu = \int f_1 d\mu \int f_2 d\mu.$$

Note that strong mixing implies mild mixing and mild mixing implies weak mixing.

In this paper we will consider the substitutions (1) $\theta : 0 \rightarrow 001, 1 \rightarrow 11001$ and (2) $\eta : 0 \rightarrow 001, 1 \rightarrow 11100$. Specifically we will investigate the mixing properties of the substitution subshifts and tilings systems arising from these substitutions. To facilitate the discussion, we present now some basic facts on substitution dynamical systems and tiling dynamical systems.

An *alphabet* \mathcal{A} is a finite set of symbols called *letters*, denoted as $\mathcal{A} = \{0, 1, 2, \dots, s-1\}$ or as letters $\mathcal{A} = \{a, b, \dots\}$. A finite string of letters is called a *word* or *block* and the set of all finite words over \mathcal{A} is denoted by $\mathcal{A}^* = \bigcup_{k \geq 1} \mathcal{A}^k$. Elements of $\mathcal{A}^{\mathbb{Z}}$ are called *sequences* over the alphabet \mathcal{A} . One can define a metric d on $\mathcal{A}^{\mathbb{Z}}$ by $d(x, y) = 2^{-n}$, where n is the smallest integer with $x_n \neq y_n$ or $x_{-n} \neq y_{-n}$ for $x = (x_n)$ and $y = (y_n)$ in $\mathcal{A}^{\mathbb{Z}}$. With this metric $\mathcal{A}^{\mathbb{Z}}$ is a compact metric space and the shift map T given by $(Tx)_n = x_{n+1}$ for a sequence $x = (x_n)$ is a homeomorphism on $\mathcal{A}^{\mathbb{Z}}$. A pair (X, T) , where X is a closed T invariant subset of $\mathcal{A}^{\mathbb{Z}}$, is called a *subshift*.

A *substitution* ζ is a map from \mathcal{A} to \mathcal{A}^* , which induces a map from \mathcal{A}^* to \mathcal{A}^* given by $\zeta(b_0 b_1 \dots b_n) = \zeta(b_0) \zeta(b_1) \dots \zeta(b_n)$. A word u is ζ -*admissible* if u occurs in $\zeta^m(a)$ for some $m \in \mathbb{N}$ and $a \in \mathcal{A}$. Define the substitution space X_ζ as the set of all sequences $x = (x_n) \in \mathcal{A}^{\mathbb{Z}}$ such that every word $x_i x_{i+1} \dots x_j$ ($i \leq j$) is ζ -admissible. A substitution ζ is called *primitive* if there exists $m \in \mathbb{N}$ such that for any $i, j \in \mathcal{A}$, i occurs in $\zeta^m(j)$. It is well-known that if a substitution ζ is primitive, then (X_ζ, T) is minimal and has a unique T -invariant probability measure μ on the Borel σ -algebra \mathcal{X} of X (see [Que]). A measure preserving system of the form $(X_\zeta, \mathcal{X}, \mu, T)$ is called a *substitution subshift* or a *substitution dynamical system*.

One dimensional substitution tiling systems were investigated by Berend and Radin ([BeRa]) and Clark and Sadun ([ClSa]). A *tile* $I = [a, b]$ in \mathbb{R} is a closed interval with the positive length $|I| = b - a > 0$. A *tiling* of \mathbb{R} is a set \mathcal{T} of tiles such that $\mathbb{R} = \bigcup \{I : I \in \mathcal{T}\}$ and distinct tiles have non-intersecting interiors. We also assume that the tiling has finitely many tiles up to translations. A finite collection of tiles in \mathcal{T} is called a \mathcal{T} -*patch* and two patches P_1 and P_2 are said to be *equivalent* if $P_2 = P_1 + t$ for some $t \in \mathbb{R}$. For a tiling \mathcal{T} of \mathbb{R} , the *tiling space* $X_{\mathcal{T}}$ is defined as the set of all tilings \mathcal{S} of \mathbb{R} with the property that every \mathcal{S} -patch is equivalent to some \mathcal{T} -patch. Define a metric on $X_{\mathcal{T}}$ as following:

$$d(\mathcal{T}_1, \mathcal{T}_2) = \min\left\{\frac{1}{\sqrt{2}}, \bar{d}(\mathcal{T}_1, \mathcal{T}_2)\right\},$$

where

$$\bar{d}(\mathcal{T}_1, \mathcal{T}_2) = \inf\{\epsilon : \text{for some } t \in \mathbb{R}^d \text{ with } |t| < \epsilon, \mathcal{T}_1 + t \text{ and } \mathcal{T}_2 \text{ agree on } B(0, 1/\epsilon)\}.$$

Then $X_{\mathcal{T}}$ is a compact space and the translation action $T_t(\mathcal{S}) = \mathcal{S} - t$ is continuous for each $t \in \mathbb{R}$.

Given a sequence $x = (x_n) \in \{0, 1, \dots, s-1\}^{\mathbb{Z}}$ and a collection of intervals $\mathcal{I} = \{J_0, J_1, \dots, J_{s-1}\}$, a *tiling* $\mathcal{T}_x := \{I_i : i \in \mathbb{Z}\}$ of \mathbb{R} is obtained by

- (1) taking I_i to be a translate of J_{x_i} ,

- (2) putting the left end point of I_0 at the origin of \mathbb{R} ,
- (3) requiring that $\mathbb{R} = \cup I_i$ and $I_i \cap I_{i+1}$ is a singleton for each $i \in \mathbb{Z}$.

A tiling \mathcal{T} is called *substitution tiling* if it is obtained from a sequence x in a substitution space X_ζ . If a substitution ζ is primitive, then there exists a unique probability measure μ on the Borel σ -algebra \mathcal{X} of the tiling space $X_\mathcal{T}$. A measure preserving system of the form $(X_\mathcal{T}, \mathcal{X}, \mu, (T_t)_{t \in \mathbb{R}})$ arising from a substitution tiling \mathcal{T} is called a *substitution tiling system*. Note that the definition of a substitution tiling space does not depend on the choice of the sequence x in the substitution space.

Dekking and Keane considered in [DeKe] substitutions (1) $\theta : 0 \rightarrow 001, 1 \rightarrow 11001$ and (2) $\eta : 0 \rightarrow 001, 1 \rightarrow 11100$ and showed that the substitution dynamical systems arising from θ and η are weakly mixing but not strongly mixing. The substitution θ is obtained by writing $a = 00$ and $b = 1$ in the Toeplitz substitution $a \rightarrow abab, b \rightarrow bbab$ ([JaKe], [Kak]) and it had been shown by Kakutani [Kak] that the symbolic dynamical system obtained by doubling the symbol a in the Toeplitz substitution is weakly mixing. Moreover, by using the method in [BeRa] and [ClSa], one can show that the tiling systems arising from these substitutions and two intervals J_0 and J_1 , where $\frac{|J_0|}{|J_1|}$ is irrational, are weakly mixing (cf. Lemma in [BeRa] and Theorem 2.5 in [ClSa]) but not strongly mixing (cf. Theorem 2.2 in [ClSa]).

These results lead to the natural question of whether these systems are mildly mixing. In this paper we will answer this question in the affirmative by showing that these systems have minimal self-joinings, which implies the mild mixing property.

The notion of *minimal self-joinings* (MSJ) was introduced by Rudolph [Ru1]. A two-fold *self-joining* of an ergodic system $\mathbf{X} = (X, \mathcal{X}, \mu, T_G)$ is a probability measure λ on $X \times X$ such that λ is invariant under the diagonal action $T_G \otimes T_G$ and the marginals of λ on each X are μ . An ergodic system $\mathbf{X} = (X, \mathcal{X}, \mu, T_G)$ is said to have minimal self-joinings of order 2 if every ergodic two-fold self-joining λ is either $\mu \times \mu$ or the image of μ under the map $x \rightarrow (x, T_g x)$ for some $g \in G$.

We denote by $C(\mathbf{X})$ the centralizer of a measure preserving system $\mathbf{X} = (X, \mathcal{X}, \mu, T_G)$, that is, the set of all measure preserving transformations commuting with T_g for all $g \in G$. It is known [Ru1] that an ergodic system with minimal self-joinings has the somewhat unusual property that it has trivial centralizer and no proper factor. Theorems 2.3 and 2.5 below show that a weak mixing measure preserving \mathbb{Z} or \mathbb{R} system with minimal self-joinings is mildly mixing.

In this paper our main result is the following.

Theorem 1.1 (cf. Theorem 3.1, 3.4, 4.1). *Let θ be the substitution $\theta(0) = 001, \theta(1) = 11001$ and let η be the substitution with $\eta(0) = 001, \eta(1) = 11100$.*

- (i) *The substitution dynamical systems associated with θ and η have minimal self-joinings, and hence are mildly mixing.*
- (ii) *The substitution tiling spaces arising from the substitutions θ and η using two intervals of irrational ratio have minimal self-joinings, and hence are mildly mixing.*

The first example of a substitution system with minimal self-joinings was Chacon's system; this was shown by del Junco, Rahe, and Swanson in [JuRaSw]. In [JuPa], del Junco and Park constructed examples of 1-dimensional \mathbb{R} -flows with minimal self-joinings based on Chacon's substitution, which can be viewed as 1-dimensional mildly mixing tiling dynamical systems. In this paper we employ methods similar to those used in [JuRaSw] and [JuPa] to prove Theorem 1.1.

Here is a brief description of the content of this paper.

In Section 2 we present basic notions and properties of joinings of dynamical systems.

In Section 3 we consider the substitution $\theta : 0 \rightarrow 001, 1 \rightarrow 11001$. First we prove that the substitution dynamical system associated with θ has minimal self-joinings and is mildly mixing. Then we show that the tiling dynamical system arising from the substitution θ and two intervals of irrational ratio has minimal self-joinings and is mildly mixing.

In Section 4 we investigate the substitution $\eta : 0 \rightarrow 001, 1 \rightarrow 11100$. We show that the substitution dynamical system associated with η and the tiling dynamical system arising from the substitution η and two intervals of irrational ratio have minimal self-joinings and are mildly mixing.

Finally, in Section 5 we present an example of a tiling dynamical system which is rigid and weakly mixing. This example is constructed based on the dynamical system investigated by del Junco and Rudolph in [JuRo].

2. JOININGS

Let $\mathbf{X} = (X, \mathcal{X}, \mu, T_G)$ and $\mathbf{Y} = (Y, \mathcal{Y}, \nu, S_G)$ be two ergodic systems. A *joining* of \mathbf{X} and \mathbf{Y} is a probability measure λ on $X \times Y$ such that (1) λ is invariant under the diagonal action $T_G \otimes S_G = (T_g \times S_g)_{g \in G}$ and (2) λ has marginals μ on X and ν on Y : $\lambda(A \times Y) = \mu(A)$ and $\lambda(X \times B) = \nu(B)$ for any $A \in \mathcal{X}$ and $B \in \mathcal{Y}$. Let $J(\mathbf{X}, \mathbf{Y})$ be the set of all joinings of \mathbf{X} and \mathbf{Y} . When $\mathbf{Y} = \mathbf{X}$, joinings are called *self-joinings (of order 2)*. In this case $J(\mathbf{X}, \mathbf{X})$ is denoted by $J(\mathbf{X})$. Given $k \in \mathbb{N}$, one can define a k -fold self-joining λ of an ergodic system \mathbf{X} as a probability measure on X^k , which is invariant under $T_G \otimes \cdots \otimes T_G$ and whose marginals on each X are μ .

The centralizer, $C(\mathbf{X})$, is the set of all invertible measure preserving maps of \mathbf{X} that commute with T_g for all $g \in G$. For $S_1, \dots, S_k \in C(\mathbf{X})$, the corresponding off-diagonal measure is the image of μ under the map $x \rightarrow (S_1(x), \dots, S_k(x))$ of X into X^k . An ergodic system $\mathbf{X} = (X, \mathcal{X}, \mu, T_G)$ is called *simple of order k* (or *k -simple*) if any k -fold ergodic self-joining λ of \mathbf{X} is a product of off-diagonal measures: there exists a partition of J_1, \dots, J_m of $\{1, 2, \dots, k\}$ such that for each J_l the projection of λ on $\prod_{i \in J_l} X$ is off-diagonal and λ is the product of these off-diagonal measures. We say that \mathbf{X} has minimal self-joinings of order k if \mathbf{X} is k -simple with $C(\mathbf{X}) = \{T_g : g \in G\}$. \mathbf{X} is called simple or has minimal self-joinings (MSJ) of all orders if it is simple or has minimal self-joinings of order k for any $k \in \mathbb{N}$ respectively.

J. King showed in ([Ki2]) that 4-fold simple systems and 4-fold minimal self-joinings systems are simple of all orders and minimal self-joinings of all orders respectively. Later Glasner, Host, and Rudolph improved King's result by showing that if (X, \mathcal{X}, μ, T) is weak-mixing and 3-simple, then it is simple.

An action $(T_g)_{g \in G}$ of a group G is called *totally weakly mixing* if each infinite subaction contains a mixing sequence (T_{t_i}) , that is, $\lim_{t_i \rightarrow \infty} \int f(x)g(T_{t_i}x) d\mu \rightarrow \int f d\mu \int g d\mu$. The following result is due to Ryzhikov.

Theorem 2.1 ([Ry] Theorem 2). *Let a 2-simple action $(T_g)_{g \in G}$ of a countable Abelian group be totally weakly mixing but not strongly mixing. In this case, $(T_g)_{g \in G}$ is simple of any order.*

If T is a weak mixing transformation, then $(T^n)_{n \in \mathbb{Z}}$ is totally weak mixing. Hence if (X, \mathcal{X}, μ, T) has two-fold minimal self-joinings and is weakly mixing, but not strongly mixing, then it has minimal self-joinings of all orders. In fact, the same result works for $G = \mathbb{R}$.

Corollary 2.1. *If $(X, \mathcal{X}, \mu, (T_t)_{t \in \mathbb{R}})$ is weakly mixing and not strongly mixing and has two-fold minimal self-joinings, then the system has minimal self-joinings of all orders.*

Proof. If $(T_t)_{t \in \mathbb{R}}$ is a weak mixing \mathbb{R} -flow on (X, \mathcal{X}, μ) , then T_α is a weak mixing transformation for any $\alpha \neq 0$. From Theorem 2.1, $\tilde{\mathbf{X}} = (X, \mathcal{X}, \mu, (T_t)_{t \in \mathbb{Q}})$ is simple of all orders and $C(\tilde{\mathbf{X}}) = \{T_t : t \in \mathbb{R}\}$. Hence $(X, \mathcal{X}, \mu, (T_t)_{t \in \mathbb{R}})$ has minimal self-joinings of all orders. \square

Now we present a mixing property of minimal self-joinings of order 2.

Theorem 2.2 (see [Gl] Corollary 12.5 and [Ru2] Theorem 6.12). *Let $\mathbf{X} = (X, \mathcal{X}, \mu, T)$ be an invertible, non-atomic dynamical system with twofold minimal self-joinings. Then,*

- (i) $C(\mathbf{X}) = \{T^n : n \in \mathbb{Z}\}$.
- (ii) (X, \mathcal{X}, μ, T) is prime, i.e., (X, \mathcal{X}, μ, T) does not have a proper factor.
- (iii) (X, \mathcal{X}, μ, T) is weakly mixing.

Theorem 2.2 (iii) can be strengthened as follows:

Theorem 2.3. *Let $\mathbf{X} = (X, \mathcal{X}, \mu, T)$ be an invertible, non-atomic dynamical system with twofold minimal self-joinings. Then it is mildly mixing.*

Proof. Suppose that $T^{n_i} f \rightarrow f$ in L^2 for some non-constant f . Then there exists a non-trivial rigid factor. From (ii) of Theorem 2.2, (X, \mathcal{X}, μ, T) is rigid. It is well-known that if (X, \mathcal{X}, μ) is non-atomic and \mathbf{X} is rigid, then $C(\mathbf{X})$ is uncountable (see [KaSiSt] or [Ki1]), which contradicts (i) of Theorem 2.2. \square

One can easily show that a result similar to Theorem 2.2 holds for \mathbb{R} -actions:

Theorem 2.4. *Let $\mathbf{X} = (X, \mathcal{X}, \mu, (T_t)_{t \in \mathbb{R}})$ be an invertible, non-atomic dynamical system with twofold minimal self-joinings. Then,*

- (i) $C(\mathbf{X}) = \{T_t : t \in \mathbb{R}\}$.

(ii) $(X, \mathcal{X}, \mu, (T_t)_{t \in \mathbb{R}})$ is prime.

It is not hard to check that the transitive flow $T_t x = x + t \bmod 1$ on $[0, 1)$ has minimal self-joinings but is not weakly mixing. Note that in this example, $(T_t)_{t \in \mathbb{R}}$ is not free; $T_n = Id$ for all $n \in \mathbb{Z}$. The following theorem shows that if the action of \mathbb{R} is free, the situation is different.

Theorem 2.5. *Let $\mathbf{X} = (X, \mathcal{X}, \mu, (T_t)_{t \in \mathbb{R}})$ be a dynamical system with minimal self-joinings. If $T_t \neq Id$ for all $t \neq 0$, then \mathbf{X} is mildly mixing.*

Proof. By Theorem 2.4 (i), the topological groups \mathbb{R} and $C(\mathbf{X})$ are isomorphic. Suppose that there exists a non-constant rigid function in $L^2(X)$. Then \mathbf{X} is rigid from Theorem 2.4 (ii). Hence $T_{t_k} \rightarrow Id$ for some increasing sequence t_k , which is impossible in \mathbb{R} . \square

3. THE SUBSTITUTION $\theta : 0 \rightarrow 001, 1 \rightarrow 11001$

3.1. The Substitution Dynamical System.

In this section we will consider the substitution dynamical system arising from the substitution θ . As we have mentioned above, it was proved in [DeKe] that this system is weakly mixing but not strongly mixing. We will strengthen this result by proving the following theorem.

Theorem 3.1. *The substitution dynamical system arising from the substitution $\theta : 0 \rightarrow 001, 1 \rightarrow 11001$ has minimal self-joinings and is mildly mixing.*

Before proving this result, let us present basic concepts and results of a primitive substitution system $(X_\zeta, \mathcal{X}, \mu, T)$ arising from a substitution ζ . Given a word $u = u_0 u_1 \cdots u_n$, let

$$[u] = [u_0 u_1 \cdots u_n] := \{x \in X_\zeta : x_0 = u_0, x_1 = u_1, \dots, x_n = u_n\}.$$

These sets and their translates are called *cylinder sets*, which are clopen and span the topology of X_ζ . It is known that if ζ is primitive, $(X_\zeta, \mathcal{X}, \mu, T)$ is uniquely ergodic ([Que]), that is, μ is the only T invariant probability measure on X_ζ . In this case we have a strong version of Ergodic Theorem:

Theorem 3.2 (see [Ox] or [Fu]). *(X, \mathcal{B}, ν, T) is uniquely ergodic if and only if for every $f \in C(X)$,*

$$\lim_{N \rightarrow \infty} \frac{1}{N} \sum_{n=1}^N f(T^n x) = \int_X f d\nu \quad (1)$$

uniformly on X .

A point $x \in X$ is called ν -generic, if x satisfies (1) for any $f \in C(X)$. If ν is ergodic measure on the topological dynamical system (X, T) , then ν -almost all points are generic with respect to ν .

If f is the characteristic function of a cylinder set for the word u in a substitution space X_ζ , then $\sum_{n=1}^N f(T^n x)$ counts the number of occurrence of u at the position i in x for $1 \leq i \leq N$. Thus the probability measure of the cylinder set $[u]$ is given by the occurrence frequency of u in x for $x \in X_\zeta$.

Let us introduce now some additional notation, which will be used throughout the Section 3.1. Define n -blocks $A_n = \theta^n(00)$ and $B_n = \theta^n(1)$ for $n \in \mathbb{N}$. Note that for any n , the finite word obtained by deleting first two letters in A_n is the same as the word obtained by deleting the first letter in B_n . Denote this word as C_n , so we have $A_n = 00C_n$ and $B_n = 1C_n$. Also notice that $A_{n+1} = A_n B_n A_n B_n$, $B_{n+1} = B_n B_n A_n B_n$, and the block C_n begins with C_{n-k} for $1 \leq k < n$. Denote the length of the word or block w by $l = l(w)$. Let l_n be the length of the word C_n , so $l_n = l(C_n)$.

The following lemma says that for each $n \in \mathbb{N}$, $x \in X_\theta$ can be uniquely written in terms of $\theta^n(00)$ and $\theta^n(1)$.

Lemma 3.1. *There is $m \in \mathbb{N}$ such that any admissible word W of X_θ with $l(W) \geq m$ has the following unique expression*

$$W = K_1 v_1 C_1 v_2 C_1 \cdots v_k C_1 K_2, \quad (2)$$

where K_1 is a suffix of $\theta(00)$ or $\theta(1)$, K_2 is a prefix of $\theta(00)$ or $\theta(1)$ and $v_i = 00$ or 1 .

Proof. It is obvious that any admissible word W can be represented by formula (2). Let us show that expression (2) is unique. Since θ is primitive there exists $m \in \mathbb{N}$ such that any admissible word $W = w_1 w_2 \cdots w_l$ with $l \geq m$ contains 11001, and so, there is i , ($1 \leq i \leq l-4$) such that $w_i w_{i+1} \cdots w_{i+4} = 11001$. If (2) is not unique, there is j with $1 \leq j \leq 3$ such that $w_{i+j} \cdots w_{i+4}$ is a prefix of $\theta(00)$ or $\theta(1)$. This is possible only for $j = 2$ and in this case the first two letters 11 of 11001 is a suffix of $\theta(00)$ or $\theta(1)$, which is a contradiction. \square

In [JuRaSw], the Structure Lemma (Lemma 1 in [JuRaSw]), obtained by the cutting and stacking method, plays a crucial role in showing that the Chacon system has minimal self-joinings. In this paper we obtain the following similar result based on Lemma 3.1.

Lemma 3.2. *Let m be a positive integer as in Lemma 3.1. If x and $y \in X_\theta$ are not in the same orbit, then for infinitely many n , there exist $m_{1,n}$ and $m_{2,n} \in \mathbb{Z}$ such that*

- (i) $|m_{i,n}| \leq (m+3)(l_n+2)$ for $i = 1, 2$ and $|m_{1,n} - m_{2,n}| \leq \frac{1}{2}(l_n+3)$,
- (ii) either $C_n 00 C_n$ occurs at $m_{1,n}$ of x and $C_n 1 C_n$ occurs at $m_{2,n}$ of y , or $C_n 1 C_n$ occurs at $m_{1,n}$ of x and $C_n 00 C_n$ occurs at $m_{2,n}$ of y .

Proof. For a given n_0 , we will find an integer $n \geq n_0$ satisfying the condition stipulated in the formulation.

Let us express x and y in terms of the n_0 -blocks $A_{n_0} = \theta^{n_0}(00)$ and $B_{n_0} = \theta^{n_0}(1)$. Then we can find an integer k ($|k| \leq \frac{1}{2}(l_{n_0}+2)$) such that the block containing x_0 begins at the same place of the block containing $(T^k y)_0$. Introduce the sequence of n_0 -blocks $(D_i^{n_0})_{i \in \mathbb{Z}}$ such that $x = \cdots D_{-1}^{n_0} D_0^{n_0} D_1^{n_0} \cdots$, where $D_i^{n_0}$ is A_{n_0} or B_{n_0} and x_0 belongs to $D_0^{n_0}$. Let δ_i be the

position where $D_i^{n_0}$ occurs in x . Similarly, $T^k y$ can be written in terms of n_0 -blocks $(E_i^{n_0})_{i \in \mathbb{Z}}$ and ϵ_i denotes the position where $E_i^{n_0}$ occurs in $T^k y$. Note that $\delta_0 = \epsilon_0$. Since x and y are not in the same orbit, there exists s such that $D_s^{n_0}$ and $E_s^{n_0}$ are different. We choose s_{n_0} such that $|s_{n_0}|$ is minimal for such s . Without loss of generality we can assume that $D_{s_{n_0}}^{n_0} = B_{n_0}$ and $E_{s_{n_0}}^{n_0} = A_{n_0}$.

Suppose $|s_{n_0}| > m$. We can express x and $T^k y$ in terms of the $(n_0 + 1)$ -blocks. Note that two different $(n_0 + 1)$ -blocks occur at the positions in x and $T^k y$ where two different n_0 -blocks $D_{s_{n_0}}^{n_0}$ and $E_{s_{n_0}}^{n_0}$ occur. Since the length of n -blocks is increasing, $|s_{n_0+1}| < |s_{n_0}|$. We can find the minimal $n \geq n_0$ such that $|s_n| \leq m$. Now we consider the two cases $s_n \geq 0$ and $s_n < 0$. If $s_n \geq 0$, then $C_n 1 C_n$ occurs at position $\delta_{s_n} - l_n$ in x and $C_n 00 C_n$ occurs at position $\delta_{s_n} - l_n$ in $T^k y$ (see figure 1.) Let $m_{1,n} = \delta_{s_n} - l_n$ and $m_{2,n} = \delta_{s_n} - l_n - k$. Then

- (a) $|m_{1,n}| \leq |\delta_{s_n}| + |l_n| \leq (m+1)(l_n+2) + l_n \leq (m+2)(l_n+2)$.
- (b) $|m_{2,n}| \leq |\delta_{s_n}| + |l_n| + |k| \leq (m+1)(l_n+2) + l_n + \frac{1}{2}(l_n+2) \leq (m+3)(l_n+2)$.
- (c) $|m_{1,n} - m_{2,n}| \leq |k| \leq \frac{1}{2}(l_n+2)$.

If $s_n < 0$, then $C_n 1 C_n$ occurs at $\delta_{s_n} - l_n$ in x and $C_n 00 C_n$ occurs at $\delta_{s_n} - l_n - 1$ in $T^k y$. Let $m_{1,n} = \delta_{s_n} - l_n$ and $m_{2,n} = \delta_{s_n} - l_n - k - 1$. Then by the similar consideration we have $|m_{i,n}| \leq (m+3)(l_n+2)$ and $|m_{1,n} - m_{2,n}| \leq |k| + 1 \leq \frac{1}{2}(l_n+3)$. \square

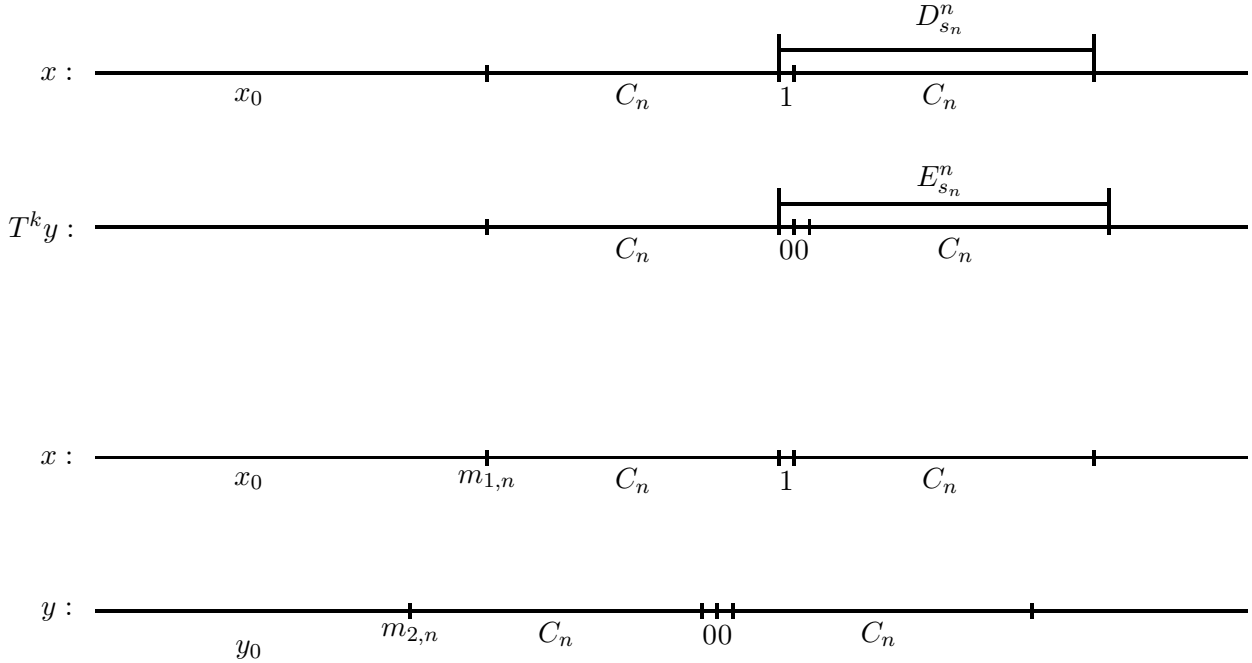


Figure 1

To show that $(X_\theta, \mathcal{X}, \mu, T)$ has minimal self-joinings, we will use the following two technical lemmas.

Lemma 3.3 ([Ru2] Lemma 6.14). *Let I be the identity map on (X, \mathcal{X}, ν_1) and S be an ergodic map on (Y, \mathcal{Y}, ν_2) . If $\bar{\nu}$ is a joining of $(X, \mathcal{X}, \nu_1, I)$ and $(Y, \mathcal{Y}, \nu_2, S)$, then $\bar{\nu} = \nu_1 \times \nu_2$.*

Lemma 3.4 ([Ru2] Lemma 6.15). *Let $\mathbf{X} = (X, \mathcal{X}, \mu, T)$ be an ergodic system and $\{P_i\}$ be a countable set of cylinder sets generating \mathcal{X} . Let $\bar{\mathcal{A}} = \{P_l \times P_m\}$ be a countable generating algebra of $X \times X$. Assume that*

- (1) $\bar{\mu} \in J(\mathbf{X})$ is a two-fold ergodic joining.
- (2) $(x, y) \in X \times X$ satisfies

$$\lim_{N \rightarrow \infty} \frac{1}{N} \sum_{n=0}^{N-1} 1_A(T^{-n}x, T^{-n}y) = \lim_{N \rightarrow \infty} \frac{1}{N} \sum_{n=0}^{N-1} 1_A(T^n x, T^n y) = \bar{\mu}(A)$$

for all $A \in \bar{\mathcal{A}}$.

- (3) There are intervals $L_k = [i_k, j_k] \subset \mathbb{Z}$, $M_k = [a_k, b_k]$, $t_k \in \mathbb{Z}$ and $\gamma > 0$ such that $i_k \leq 0 \leq j_k$ and $j_k - i_k \rightarrow \infty$, $M_k \subset L_k$ and $t_k + M_k \subset L_k$, and $|M_k| \geq \gamma |L_k|$.
- (4) For any cylinder sets P_l, P_m , there exists $K = K(P_l, P_m)$ such that if $k \geq K$, then for all $i \in M_k$,

$$\begin{aligned} T^i x \in P_l & \text{ if and only if } T^{t_k+i} x \in P_l, \\ T^i y \in P_m & \text{ if and only if } T^{t_k+i+1} y \in P_m. \end{aligned}$$

Then $\bar{\mu}$ is an $I \times T$ invariant measure on $X \times X$ and hence $\bar{\mu} = \mu \times \mu$.

Proof of Theorem 3.1. Note that the substitution dynamical system $(X_\theta, \mathcal{X}, \mu, T)$ arising from θ is weakly mixing, hence μ is non-atomic. We know from [DeKe] that it is not strongly mixing. Hence if we show $(X_\theta, \mathcal{X}, \mu, T)$ has two-fold minimal self-joinings, then it will have minimal self-joinings of all orders by Corollary 2.1 and will be mildly mixing by Theorem 2.3.

Given an ergodic joining $\bar{\mu}$, we can find a $\bar{\mu}$ -generic point (x, y) satisfying condition (2) in Lemma 3.4. If x and y are in the same orbit, that is $T^k x = y$ for some $k \in \mathbb{Z}$, then $\bar{\mu}$ is an off-diagonal measure which is the image of μ under the map $z \rightarrow (z, T^k z)$.

Otherwise, introduce a new alphabet \mathcal{A}_s consisting of the letters h_i^s , where $C_s = h_1^s \cdots h_{l_s}^s$ for each s . (Recall that C_s is a block such that $\theta^s(00) = 00C_s$.) Let P_i^s be the cylinder for the alphabet h_i^s . Then $\{P_i^s : 1 \leq s \leq l_s, i \in \mathbb{N}\}$ is a countable set of cylinder sets generating \mathcal{X} .

By Lemma 3.2, without loss of generality we assume that there exists an increasing sequence of integers $(n_k)_{k=1}^\infty$ such that the two blocks $C_{n_k} 1C_{n_k}$ and $C_{n_k} 00C_{n_k}$ appear in $L_k = [-(m+4)(l_{n_k}+2), (m+4)(l_{n_k}+2)]$ of x and y respectively (see figure 2). Let M_k be the interval in the first C_{n_k} blocks where $C_{n_k} 1C_{n_k}$ and $C_{n_k} 00C_{n_k}$ overlap. Its length satisfies $|M_k| \geq |C_{n_k}| - |m_{1,n_k} - m_{2,n_k}| \geq \frac{1}{2}l_{n_k} - 3$. Let $t_k = l_{n_k} + 1$. Then for $s \geq n_k$, we have

$$T^i x \in P_l^s \Leftrightarrow T^{t_k+i} x \in P_l^s \quad \text{and} \quad T^i y \in P_m^s \Leftrightarrow T^{t_k+i+1} y \in P_m^s$$

for any $1 \leq l, m \leq h_{l_s}$. Also, we have

$$\liminf_{k \rightarrow \infty} \frac{|M_k|}{2(m+4)(l_{n_k}+2)} \geq \liminf_{k \rightarrow \infty} \frac{l_{n_k} - 6}{4(m+4)(l_{n_k}+2)} = \frac{1}{4(m+4)} > 0.$$

Lemma 3.4 implies that $\bar{\mu}$ is $I \times T$ invariant and thus, $\bar{\mu} = \mu \times \mu$. \square

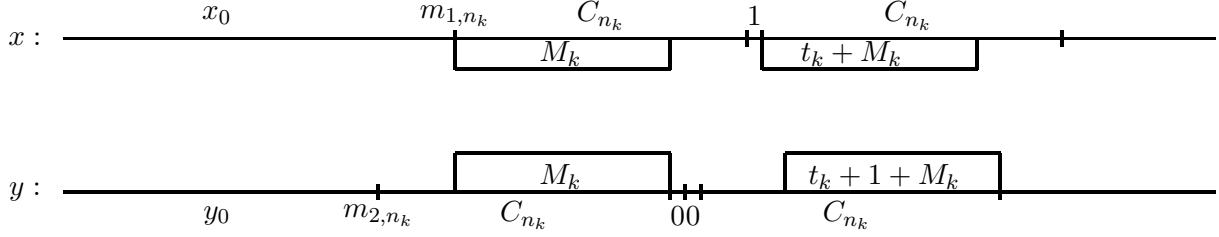


Figure 2

3.2. Tiling Spaces.

Let us present some basic facts on the substitution tiling system arising from the substitution ζ on the alphabet $\{0, 1\}$ and intervals J_0 and J_1 . Let \mathcal{T} be a substitution tiling arising from ζ and intervals J_0 and J_1 and let $X_{\mathcal{T}}$ be the corresponding tiling space. For any ζ -admissible word $u = u_0 u_1 \cdots u_n$ and any interval $I \subset [0, |J_{u_0}|)$, define

$$[u] \times I := \{\mathcal{S} \in X_{\mathcal{T}} : \text{for some } t \in I, (\mathcal{S} - t) \text{ and } J_{u_0} J_{u_1} \cdots J_{u_n} \text{ agree on } [0, |J_{u_0}| + \cdots + |J_{u_n}|)\}.$$

These sets and their translates are called cylinder sets in the tiling space. They are clopen sets and span the topology of the tiling space. Another description of the \mathbb{R} -flow on the substitution tiling space is as a flow under the function f built over the substitution subshift (X_{ζ}, T) , where $f : X_{\zeta} \rightarrow \mathbb{R}$ is defined by $f(x) = |J_0|$ for $x \in [0]$ and $f(x) = |J_1|$ for $x \in [1]$.

It is known that the tiling system is minimal and has a unique invariant probability measure ν if the substitution is primitive. Then we have

$$\nu([u] \times I) = \mu([u]) \times \frac{|I|}{\mu([0])|J_0| + \mu([1])|J_1|}, \quad (3)$$

where μ is the unique invariant probability measure on the substitution system arising from ζ (see Lemma 2.1 in [ClSa]). Clark and Sadun obtained the following condition for this class of tiling systems to be weakly mixing.

Theorem 3.3 (cf [ClSa] Theorem 2.5). *Let ζ be an aperiodic, primitive, constant length substitution on the alphabet $\mathcal{A} = \{0, 1\}$, where the number of 0s occurring in $\zeta(0)$ is different from the number of 0s occurring in $\zeta(1)$. Then the substitution tiling system arising from ζ and intervals J_0 and J_1 , where $\frac{|J_0|}{|J_1|}$ is irrational, is weakly mixing.*

See [Ro] and [So] for more information about tiling dynamical systems and [ClSa] for 1-dimensional substitution tiling spaces.

In this subsection we will consider the substitution tiling \mathcal{T} of \mathbb{R} arising from the substitution $\theta : 0 \rightarrow 001, 1 \rightarrow 11001$ and two intervals J_0, J_1 and the corresponding substitution tiling system $\mathbf{X}_{\mathcal{T}} = (X_{\mathcal{T}}, \mathcal{D}, \lambda, (T_t)_{t \in \mathbb{R}})$. Our goal is to show that if $\frac{|J_0|}{|J_1|}$ is irrational, then the tiling system $\mathbf{X}_{\mathcal{T}}$ has minimal self-joinings and is mildly mixing.

By writing $a = 00$ and $b = 1$, we obtain a substitution $\tilde{\theta} : a \rightarrow abab, b \rightarrow bbab$ from the substitution θ . Let $\tilde{\mathcal{T}}$ be the substitution tiling arising from the substitution $\tilde{\theta}$ and intervals J_a and J_b , and let $\mathbf{X}_{\tilde{\mathcal{T}}} = (X_{\tilde{\mathcal{T}}}, \tilde{\mathcal{D}}, \tilde{\lambda}, (T_t)_{t \in \mathbb{R}})$ be the corresponding substitution tiling system. If $|J_a| = 2|J_0|$ and $|J_b| = |J_1|$, then any tiling in $\mathbf{X}_{\mathcal{T}}$ uniquely corresponds to a tiling in $\mathbf{X}_{\tilde{\mathcal{T}}}$ by writing $J_0 J_0 = J_a$ and $J_1 = J_b$. From this observation, it is not hard to see that $\mathbf{X}_{\tilde{\mathcal{T}}}$ and $\mathbf{X}_{\mathcal{T}}$ are topologically conjugate if $|J_a| = 2|J_0|$ and $|J_b| = |J_1|$. Since these systems have unique invariant probability measures, they are also measurably isomorphic. Our main result in this section is the following. (Note that by Theorem 3.3 the systems $\mathbf{X}_{\mathcal{T}}$ and $\mathbf{X}_{\tilde{\mathcal{T}}}$ appearing in Theorem 3.4 are weakly mixing.)

Theorem 3.4. *If $\frac{|J_a|}{|J_b|}$ is irrational, then $\mathbf{X}_{\tilde{\mathcal{T}}} = (X_{\tilde{\mathcal{T}}}, \tilde{\mathcal{D}}, \tilde{\lambda}, (T_t)_{t \in \mathbb{R}})$ has minimal self-joinings and is mildly mixing. Hence if $\frac{|J_0|}{|J_1|}$ is irrational, $\mathbf{X}_{\mathcal{T}}$ has minimal self-joinings and is mildly mixing.*

The following lemma is the \mathbb{R} -flow version of Lemma 3.4. It is not hard to prove Lemma 3.5 using the argument in the proof of Lemma 6.15 in [Ru2].

Lemma 3.5. *Let $\mathbf{X} = (X, \mathcal{X}, \mu, (T_t)_{t \in \mathbb{R}})$ be an ergodic system. Let $\alpha \neq 0$ such that T_{α} is an ergodic transformation. Let $\{P_i\}$ be a countable set of cylinder sets generating \mathcal{X} . Let $\overline{\mathcal{A}} = \{P_l \times P_m\}$ be a countable generating algebra of $X \times X$. Assume that*

- (1) $\bar{\mu} \in J(\mathbf{X})$ is a two-fold ergodic joining.
- (2) $(x, y) \in X \times X$ satisfies

$$\lim_{L \rightarrow \infty} \frac{1}{L} \int_{-L}^0 1_A(T_t x, T_t y) dt = \lim_{L \rightarrow \infty} \frac{1}{L} \int_0^L 1_A(T_t x, T_t y) dt = \bar{\mu}(A)$$

for all $A \in \overline{\mathcal{A}}$.

- (3) There are intervals $L_k = [i_k, j_k] \subset \mathbb{R}$, $M_k = [a_k, b_k]$, $t_k \in \mathbb{R}$ and $\gamma > 0$ such that $i_k \leq 0 \leq j_k$ and $j_k - i_k \rightarrow \infty$, $M_k \subset L_k$ and $t_k + M_k \subset L_k$, and $|M_k| \geq \gamma |L_k|$.
- (4) For any cylinder sets P_l, P_m , there exists $K = K(P_l, P_m)$ such that if $k \geq K$, then for all $t \in M_k$,

$$\begin{aligned} T_t x &\in P_l \text{ if and only if } T_{t_k+t} x \in P_l, \\ T_t y &\in P_m \text{ if and only if } T_{t_k+t+\alpha} y \in P_m. \end{aligned}$$

Then $\bar{\mu}$ is an $I \times T_{\alpha}$ invariant measure on $X \times X$ and hence $\bar{\mu} = \mu \times \mu$.

Proof of Theorem 3.4. We will assume that $|J_b| > |J_a|$. The proof for the other case is analogous. For $\alpha = |J_b| - |J_a|$, T_{α} is a weak-mixing transformation on $(X_{\tilde{\mathcal{T}}}, \tilde{\mathcal{D}}, \tilde{\lambda})$, since $\mathbf{X}_{\tilde{\mathcal{T}}}$ is weakly mixing.

Let $\bar{\lambda}$ be a twofold ergodic joining of $\mathbf{X}_{\tilde{\gamma}}$ and let $(\mathcal{S}_1, \mathcal{S}_2)$ be a $\bar{\lambda}$ -generic point satisfying the condition (2) in Lemma 3.5.

If \mathcal{S}_1 and \mathcal{S}_2 are on the same orbit under $(T_t)_{t \in \mathbb{R}}$, then $\bar{\lambda}$ is off-diagonal. Otherwise, we can view \mathcal{S}_1 and \mathcal{S}_2 as the tilings arising from tiles J_0 and J_1 by writing $J_a = J_0 J_0$ and $J_b = J_1$. Then, for infinitely many n , patches corresponding to $C_n 1 C_n$ and $C_n 00 C_n$ occurs in \mathcal{S}_1 and \mathcal{S}_2 such that they occur not too far from the origin and their overlap is sufficiently large. By applying the same argument as in the proof of Theorem 3.1 with Lemma 3.5, $\bar{\lambda}$ is $I \times T_\alpha$ invariant, so $\bar{\lambda} = \tilde{\lambda} \times \tilde{\lambda}$. Hence $\mathbf{X}_{\tilde{\gamma}}$ has two-fold minimal self-joinings, and so it is mildly mixing. Moreover, it has minimal self-joinings of all orders since the substitution tiling system is not strongly mixing. \square

3.3. More examples.

The previous results can be generalized. Let s be a primitive substitution of constant length on the alphabet $\{a, b\}$ with $s(a) = aA$ and $s(b) = bA$, where A is a finite word over $\{a, b\}$. In this subsection we will show that a substitution tiling system arising from s and two intervals J_a, J_b , where $\frac{|J_a|}{|J_b|}$ is irrational, has minimal self-joinings and so is mildly mixing.

Let us first show that this systems is weakly mixing. By Theorem 3.3, it is enough to show that s is an aperiodic substitution. Suppose that $T^k(x) = x$ for some $x \in X_s$. There is a non-negative integer n such that $l(s^n(a)) \leq k < l(s^{n+1}(a))$. Let A_n be the block such that $s^n(a) = aA_n$ and $s^n(b) = bA_n$. Note that $A = A_1$. The blocks aA_n and bA_n both appear in x . Since $T^k(x) = x$, the k -th letter in A_n is a from aA_n and the k -th letter in A_n is b from bA_n , which leads to a contradiction. Hence, a tiling system arising from the substitution s and two intervals J_a, J_b , where $\frac{|J_a|}{|J_b|}$ is irrational, is weakly mixing. We have the following lemma similar to Lemma 3.2.

Lemma 3.6. *There is $m \in \mathbb{N}$ such that any admissible word W of X_s with $l(W) \geq m$ has the following unique expression*

$$W = K_1 v_1 A v_2 A \cdots v_k A K_2,$$

where K_1 is a suffix of $s(a)$ or $s(b)$, K_2 is a prefix of $s(a)$ or $s(b)$ and $v_i = a$ or b .

Proof. Since s is primitive, aa or bb is an admissible word. We may assume that bb is admissible. Then bba is also admissible, otherwise there exists a sequence $x = (x_n) \in X_s$ such that $x_n = b$ for all n .

Choose m such that any admissible word $W = w_1 w_2 \cdots w_l$ with $l \geq m$ contains $s(bba) = bAbAaA$. It is clear that we have the above expression for W . If there is another expression, then $\alpha A \beta A$ occurs in $bAbAaA$, where α occurs in the first A of $bAbAaA$ and β occurs in the second A of $bAbAaA$. Then the letter b occurs at some position i of the first A of $\alpha A \beta A$ and the letter a occurs in the same position i of the second A of $\alpha A \beta A$, which is a contradiction. \square

Let (X, \mathcal{X}, μ, T) be a subshift generated by $x \in \{0, 1\}^{\mathbb{Z}}$, where x is obtained from a sequence $y \in X_s$ by writing $a = 00$ and $b = 1$. Let C_n be the block of letters 0 and 1 obtained from

A_n by writing $a = 00$ and $b = 1$. Applying Lemma 3.6, we can obtain a result similar to Lemma 3.2 for the subshift (X, \mathcal{X}, μ, T) with an appropriate m . The methods used in Sections 3.1 and 3.2 imply that (X, \mathcal{X}, μ, T) has minimal self-joinings and that the tiling dynamical system arising from the substitution s and intervals J_0, J_1 , where $\frac{|J_0|}{|J_1|}$ is irrational, has minimal self-joinings.

4. THE SUBSTITUTION $\eta : 0 \rightarrow 001, 1 \rightarrow 11100$

Let $(X_\eta, \mathcal{X}, \mu, T)$ be the dynamical system arising from the substitution η and two intervals J_0, J_1 of irrational ratio. Let $\tilde{\eta}$ be the substitution $a \rightarrow abab, b \rightarrow bbba$. Then the substitution tiling system arising from η and two intervals J_0, J_1 is isomorphic to the substitution tiling system arising from $\tilde{\eta}$ and two intervals J_a, J_b if $|J_a| = 2|J_0|$ and $|J_b| = |J_1|$. In [BeRa] it was shown that the substitution tiling system arising from $\tilde{\eta}$ and two intervals J_a, J_b , where $\frac{|J_a|}{|J_b|}$ is irrational, is weakly mixing. Thus, the substitution tiling system arising from η and two intervals J_0, J_1 , where $\frac{|J_0|}{|J_1|}$ is irrational, is weakly mixing. In this section we will strengthen these results.

Theorem 4.1. *Let η be a substitution with $\eta(0) = 001$ and $\eta(1) = 11100$.*

- (a) *The substitution dynamical system $(X_\eta, \mathcal{X}, \mu, T)$ has minimal self-joinings and is mildly mixing.*
- (b) *The substitution tiling system arising from η and intervals J_0 and J_1 , where $\frac{|J_0|}{|J_1|}$ is irrational, has minimal self-joinings and is mildly mixing.*

Define n -blocks $A_n = \eta^n(00)$ and $B_n = \eta^n(1)$. Denote $l_n = l(A_n) = l(B_n) + 1$. Note that $l_{n+1} = 4l_n + 1$. The following lemma is similar to Lemma 3.1.

Lemma 4.1. *There is $m \in \mathbb{N}$ such that any admissible word W of X_η with $l(W) \geq m$ has the following unique expression*

$$W = K_1 v_1 C_1 v_2 C_2 \cdots v_k C_k K_2,$$

where K_1 is a suffix of $\eta(00)$ or $\eta(1)$, K_2 is a prefix of $\eta(00)$ or $\eta(1)$, and $v_i C_i$ is $\eta(00)$ or $\eta(1)$.

The proof for this lemma is just the proof of Lemma 3.1, using the word 11100 instead of 11001. From this result, we have the following structure lemma.

Lemma 4.2. *Let m be the positive integer in Lemma 4.1. If x and y are not in the same orbit, then for infinitely many n , there exists m_1 and m_2 with $|m_i| \leq (m + 3)l_{n+1}$ and $|m_1 - m_2| \leq \frac{1}{2}l_n + 2$ such that one of the following occurs:*

- (i) $A_n B_n A_n$ and $B_n B_n B_n$ occur at x_{m_1} and y_{m_2} (or y_{m_2} and x_{m_1}) respectively.
- (ii) $B_n A_n B_n$ and $B_n B_n B_n$ occur at x_{m_1} and y_{m_2} (or y_{m_2} and x_{m_1}) respectively.
- (iii) $A_n A_n$ and $B_n B_n$ occur at x_{m_1} and y_{m_2} (or y_{m_2} and x_{m_1}) respectively.

Proof. Given $n_0 \in \mathbb{N}$, x and y can be expressed in terms of $A_{n_0+1} = \eta^{n_0+1}(00)$ and $B_{n_0+1} = \eta^{n_0+1}(1)$. Then we can find an integer \tilde{k} ($|\tilde{k}| \leq \frac{1}{2}l_{n_0+1}$) such that the block containing x_0 begins at the same place of the block containing $(T^{\tilde{k}}y)_0$. Introduce the sequence of $(n_0 + 1)$ -blocks $(D_i^{n_0+1})_{i \in \mathbb{Z}}$ such that $x = \dots D_{-1}^{n_0+1} D_0^{n_0+1} D_1^{n_0+1} \dots$, where $D_i^{n_0+1}$ is A_{n_0+1} or B_{n_0+1} and x_0 belongs to $D_0^{n_0+1}$. Let δ_i be the position where $D_i^{n_0+1}$ occurs in x . Similarly, write $T^{\tilde{k}}y$ in terms of the $(n_0 + 1)$ -blocks $(E_i^{n_0+1})_{i \in \mathbb{Z}}$ and let ϵ_i denote the position where $E_i^{n_0+1}$ occurs in $T^{\tilde{k}}y$. Since x and y are not in the same orbit, there exists s such that $D_s^{n_0+1}$ and $E_s^{n_0+1}$ are different. We choose s_{n_0+1} such that $|s_{n_0+1}|$ is minimal for such s . If $|s_{n_0+1}| > m + 1$, we express x and $T^{\tilde{k}}y$ as sequences of $(D_i^{n_0+2})$ and $(E_i^{n_0+2})$. In this case, $|s_{n_0+2}| < |s_{n_0+1}|$. Hence we can find an integer $n \geq n_0$ such that $|s_{n+1}| \leq m$.

Let $k = \tilde{k}$ if $s_{n+1} \geq 0$. If $s_{n+1} < 0$, then choose $k = \tilde{k} \pm 1$ such that the two different $(n + 1)$ -blocks in x and $T^k y$ occur at the same place. Hence, one of the following holds:

- (1) A_{n+1} occurs in x and B_{n+1} occurs in $T^k y$ at the same place i with $|i| \leq (m + 1)l_{n+1}$,
- (2) B_{n+1} occurs in x and A_{n+1} occurs in $T^k y$ at the same place i with $|i| \leq (m + 1)l_{n+1}$.

Without loss of generality we can assume that (1) holds. Recall that $A_{n+1} = A_n B_n A_n B_n$, $B_{n+1} = B_n B_n B_n A_n$, and $l_{n+1} = 4l_n - 2$ (see figure 3.)

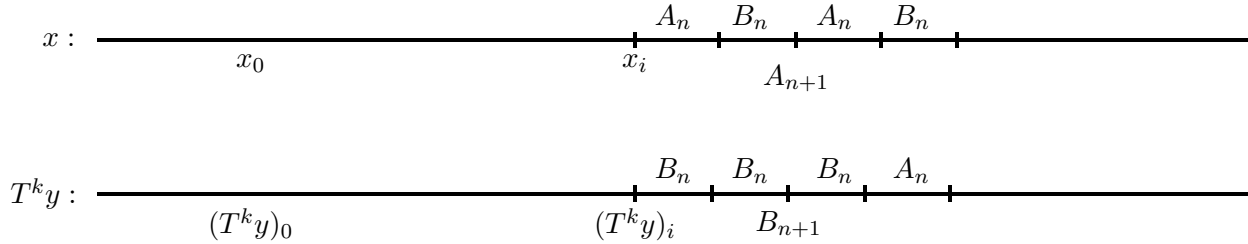


Figure 3

Let us consider the following 5 cases for $|k| \leq \frac{1}{2}l_{n+1} + 1$:

- (a) $-\frac{1}{8}l_{n+1} \leq k < \frac{1}{8}l_{n+1}$ ($\Rightarrow |k| \leq \frac{1}{8}l_{n+1} \leq \frac{1}{2}l_n$): Then $A_n B_n A_n$ occurs at $m_1 = i$ in x and $B_n B_n B_n$ occurs at $m_2 = i - k$ in y (see figure 4). We have $|m_1| \leq (|s_{n+1}| + 1)l_{n+1} \leq (m + 1)l_{n+1}$, $|m_2| \leq (|s_{n+1}| + 1)l_{n+1} + |k| \leq (m + 1)l_{n+1} + \frac{1}{8}l_{n+1} \leq (m + 2)l_{n+1}$, and $|m_1 - m_2| = |k| \leq \frac{1}{2}l_n$, so (i) holds.

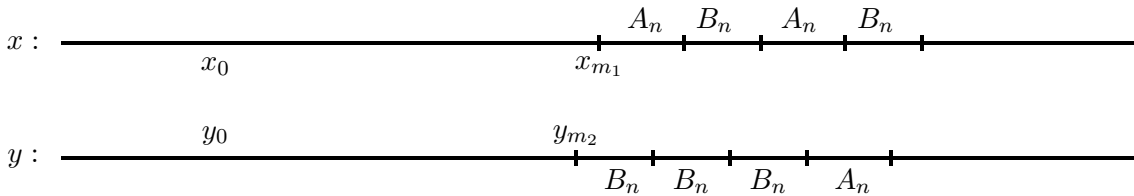


Figure 4

- (b) $-\frac{3}{8}l_{n+1} \leq k < -\frac{1}{8}l_{n+1}$ ($\Rightarrow -\frac{3}{2}l_n + \frac{3}{4} \leq k < -\frac{1}{2}l_n + \frac{1}{4}$): Then $B_n A_n B_n$ occurs at $m_1 = i + l_n$ in x and $B_n B_n B_n$ occurs at $m_2 = i - k$ in y (see figure 5). We have $|m_1| \leq (m+1)l_{n+1} + l_n \leq (m+2)l_{n+1}$ and $|m_2| \leq (m+1)l_{n+1} + \frac{3}{8}l_{n+1} \leq (m+2)l_{n+1}$. Also, $m_1 - m_2 = l_n + k$ and $-\frac{1}{2}l_n + \frac{3}{4} \leq l_n + k < \frac{1}{2}l_n + \frac{1}{4}$, so $|m_1 - m_2| \leq \frac{1}{2}l_n + 1$, so (ii) holds.

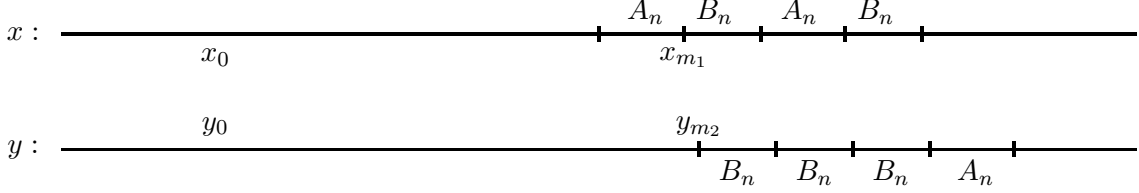


Figure 5

- (c) $-\frac{1}{2}l_{n+1} - 1 \leq k < -\frac{3}{8}l_{n+1}$ ($\Rightarrow -2l_n \leq k < -\frac{3}{2}l_n + \frac{3}{4}$): Then $B_{n+1} = B_n B_n B_n A_n$ occurs at $i - k$ in y (see figure 6 (i)).

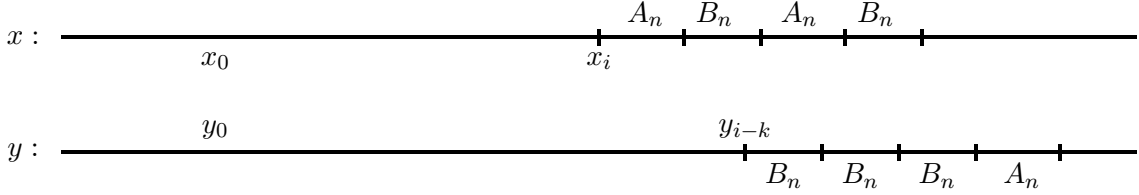


Figure 6 (i)

Now consider the blocks occurring before the block $B_{n+1} = B_n B_n B_n A_n$ at $i - k$ in y . There are two possibilities: either $A_{n+1} = A_n B_n A_n B_n$ occurs right before B_{n+1} or $B_{n+1} = B_n B_n B_n A_n$ occurs right before B_{n+1} in y .

For the first case, B_n occurs at $m_2 = i - k - l_n + 1$ in y . Note that $B_n A_n B_n$ occurs at $m_1 = i + l_n$ in x and $B_n B_n B_n$ occurs at $m_2 = i - k - l_n + 1$ in y (see figure 6 (ii)). Then $|m_1| \leq (m+1)l_{n+1} + l_n \leq (m+2)l_{n+1}$ and $|m_2| \leq (m+1)l_{n+1} + (\frac{1}{2}l_{n+1} + 1) + l_n + 1 \leq (m+2)l_{n+1}$. Also $m_1 - m_2 = k + 2l_n - 1$. Then $-1 \leq k + 2l_n - 1 < \frac{1}{2}l_n - \frac{1}{4}$, so $|m_1 - m_2| \leq \frac{1}{2}l_n + 2$, so (iii) holds.

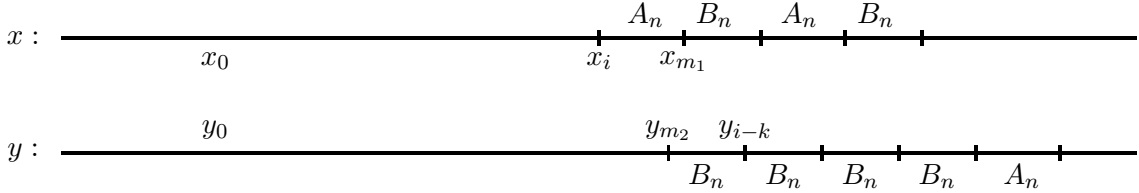


Figure 6 (ii)

Otherwise $B_n B_n B_n A_n$ occurs at $i - k - 4l_n - 1$ of y . Let us consider the blocks occurring before $A_{n+1} = A_n B_n A_n B_n$ at i in x ; either $B_{n+1} = B_n B_n B_n A_n$ or $A_{n+1} = A_n B_n A_n B_n$ occurs before A_{n+1} at i in x .

For the first case A_n occurs at $i - l_n$ of x . Then $A_n A_n$ occurs at $m_1 = i - l_n$ in x and $B_n B_n$ occurs at $m_2 = i - k - 3l_n + 2$ of y (see figure 6 (iii)). We have $|m_1| \leq (m+1)l_{n+1} + l_n \leq (m+2)l_{n+1}$ and $|m_2| \leq (m+1)l_{n+1} + (\frac{1}{2}l_{n+1} + 1) + 3l_n + 2 \leq (m+3)l_{n+1}$. Also $m_1 - m_2 = k + 2l_n - 2$, so $|m_1 - m_2| \leq \frac{1}{2}l_n + 2$, so (iii) holds.

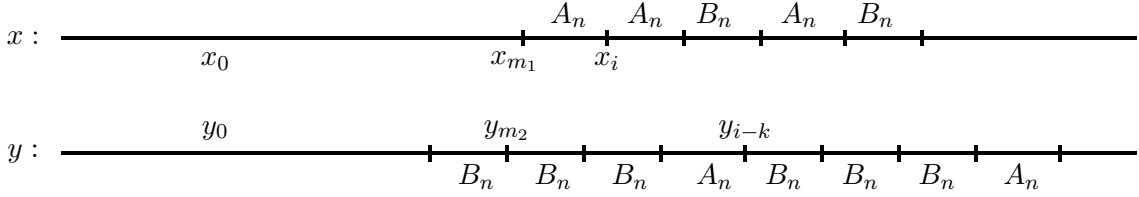


Figure 6 (iii)

For the latter case $A_n B_n A_n$ occurs at $m_1 = i - 2l_n + 1$ in x and $B_n B_n B_n$ occurs at $m_2 = i - k - 4l_n + 3$ in y (see figure 6 (iv)). We have $|m_1| \leq (m+1)l_{n+1} + 2l_n + 1 \leq (m+2)l_{n+1}$ and $|m_2| \leq (m+1)l_{n+1} + \frac{1}{2}l_{n+1} + 1 + 4l_n + 3 \leq (m+3)l_{n+1}$. Also $m_1 - m_2 = k + 2l_n - 2$, so $|m_1 - m_2| \leq \frac{1}{2}l_n + 2$, so (i) holds.

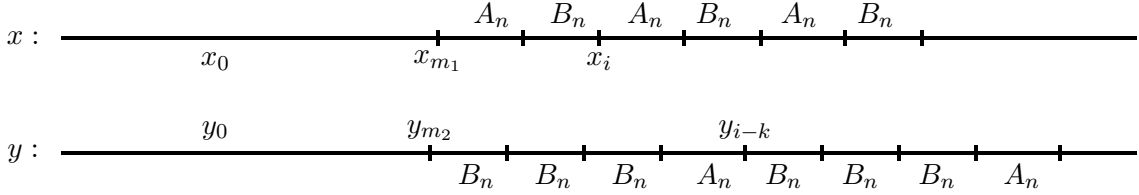


Figure 6 (iv)

- (d) $\frac{1}{8}l_{n+1} \leq k < \frac{3}{8}l_{n+1}$ ($\Rightarrow \frac{1}{2}l_n - \frac{1}{4} \leq k < \frac{3}{2}l_n - \frac{3}{4}$): $A_{n+1} = A_n B_n A_n B_n$ occurs at i in x . We have A_n or B_n right before this block A_{n+1} in x . For the first case, $A_n A_n$ occurs at $m_1 = i - l_n$ in x and $B_n B_n$ occurs at $m_2 = i - k$ in y (see the figure 7 (i)). Then $|m_i| \leq (m+2)l_{n+1}$ for $i = 1, 2$ and $m_2 - m_1 = l_n - k$. Since $-\frac{1}{2}l_n + \frac{3}{4} < l_n - k \leq \frac{1}{2}l_n + \frac{1}{4}$, we have $|m_1 - m_2| \leq \frac{1}{2}l_n + 2$, so (iii) holds.

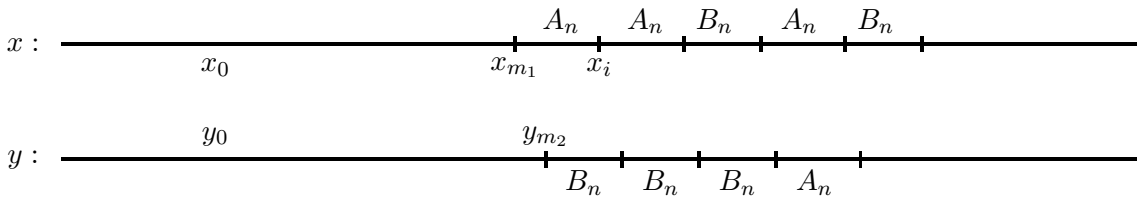


Figure 7 (i)

For the latter case, $B_n A_n B_n$ occurs at $m_1 = i - l_n + 1$ in x and $B_n B_n B_n$ occurs at $m_2 = i - k$ in y (see figure 7 (ii)). Similarly, we obtain $|m_i| \leq (m + 2)l_{n+1}$ for $i = 1, 2$ and $|m_1 - m_2| \leq \frac{1}{2}l_n + 2$, so (ii) holds.

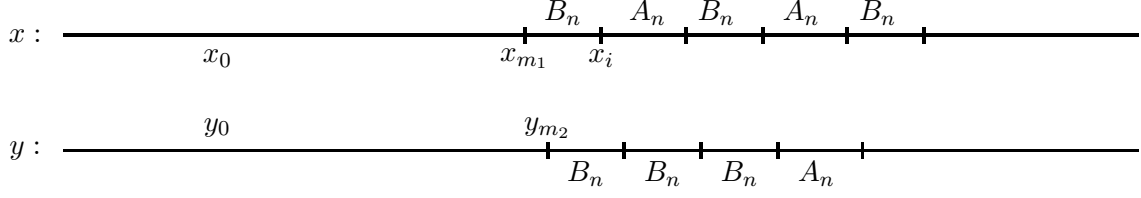


Figure 7 (ii)

- (e) $\frac{3}{8}l_{n+1} \leq k \leq \frac{1}{2}l_{n+1}$ ($\Rightarrow \frac{3}{2}l_n - \frac{3}{4} \leq k \leq 2l_n - 1$): $A_{n+1} = A_n B_n A_n B_n$ occurs at i in x . Then either B_{n+1} or A_{n+1} occurs before this A_{n+1} . For the first case, $A_n A_n$ occurs at $m_1 = i - l_n$ in x and $B_n B_n$ occurs at $m_2 = i - k + l_n - 1$ in y (see figure 8 (i)). Then, we have $|m_i| \leq (m + 2)l_{n+1}$ and $m_2 - m_1 = -k + 2l_n - 1$, so $|m_1 - m_2| \leq \frac{1}{2}l_n + 2$, so (iii) holds.

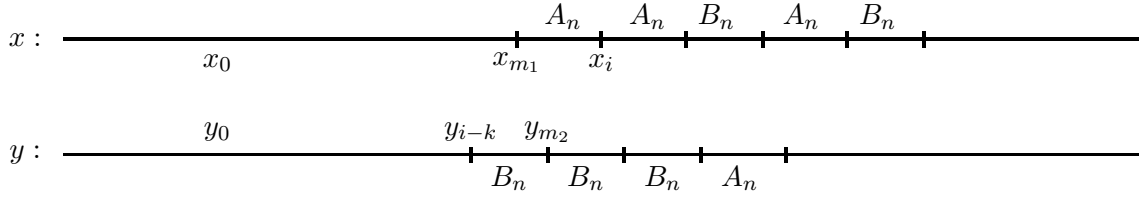


Figure 8 (i)

For the latter case $A_n B_n A_n$ occurs at $m_1 = i - 2l_n + 1$ in x and $B_n B_n B_n$ occurs at $m_2 = i - k$ in y (see figure 8 (ii)). Then $|m_i| \leq (m + 2)l_{n+1}$ for $i = 1, 2$ and $|m_1 - m_2| \leq \frac{1}{2}l_n + 2$, so (i) holds.

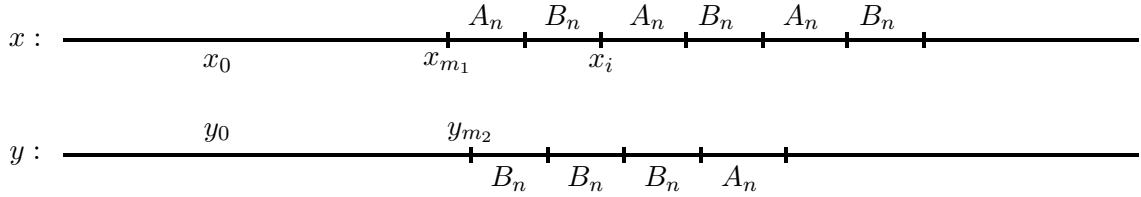


Figure 8 (ii)

□

Proof of Theorem 4.1. (a) The proof is similar to the proof of Theorem 3.1. It is enough to show that $(X_\eta, \mathcal{X}, \mu, T)$ has minimal self-joinings of order 2. Let $\bar{\mu}$ be a twofold ergodic joining of X_η and let (x, y) be a $\bar{\mu}$ -generic point. If x and y are on the same orbit under T , then $\bar{\mu}$ is off-diagonal.

Otherwise there exists an increasing sequence (n_k) such that at least one of the conclusions (i) – (iii) of Lemma 4.2 occurs. Suppose that the conclusion of (i) of Lemma 4.2 occurs for a sequence (n_k) . We assume that $A_{n_k}B_{n_k}A_{n_k}$ occurs at x_{m_1} and $B_{n_k}B_{n_k}B_{n_k}$ occurs at y_{m_2} (see figure 9).

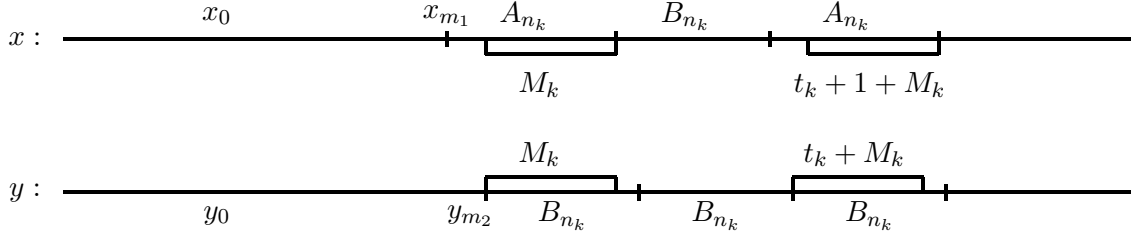


Figure 9

Let $L_k = [-(m+4)l_{n_k+1}, (m+4)l_{n_k+1}]$ and let M_k be the interval where the first A_{n_k} of $A_{n_k}B_{n_k}A_{n_k}$ and the first B_{n_k} of $B_{n_k}B_{n_k}B_{n_k}$ overlap. Choose $t_k = 2l_{n_k} - 2$. Applying the argument in Theorem 3.1 with Lemma 3.4, we have that $\bar{\mu}$ is $T \times I$ invariant, so $\bar{\mu} = \mu \times \mu$. If one of the conclusions (ii) and (iii) of Lemma 4.2 holds for a sequence (n_k) , choose t_k as (ii) $2l_{n_k} - 2$, or (iii) $l_{n_k} - 1$ according the conclusion of Lemma 4.2 and apply Lemma 3.4, we have that $\bar{\mu}$ is $T \times I$ invariant or $I \times T$ invariant, so $\bar{\mu} = \mu \times \mu$.

(b) The proof of (b) follows along the lines of the proof of Theorem 3.4, combined with Lemma 4.2, and is omitted. \square

5. RIGIDITY AND WEAK MIXING

In this section we construct an example of a tiling dynamical system which is rigid and weakly mixing. Recall that $(X, \mathcal{X}, \mu, T_G)$ is said to be rigid if there is a sequence (t_i) in G with $t_i \rightarrow \infty$ such that for every $f \in L^2(X)$, $f \circ T_{t_i}$ converges to f in $L^2(X)$, which is equivalent to the fact that $\mu(A \cap T_{t_i}A) \rightarrow \mu(A)$ for any $A \in \mathcal{X}$.

Let us consider first the following subshift, which was investigated in [JuRo]. A sequence of blocks B_n is defined as follows:

$$B_1 = 010, B_{n+1} = B_n^{2^n} 1 B_n^{2^n}.$$

Let $X = \{x = (x_n) \in \{0, 1\}^{\mathbb{Z}} : \text{for all } i < j, x_i x_{i+1} \cdots x_j \text{ occurs in some } B_k\}$. Then X is a closed shift invariant subset of $\{0, 1\}^{\mathbb{Z}}$. It is known that (X, T) is minimal and uniquely ergodic. Moreover, equipped with the shift invariant probability measure μ on the Borel

σ -algebra \mathcal{X} , the measure preserving system (X, \mathcal{X}, μ, T) is rigid and weakly mixing (see [JuRo].)

Now let us consider a tiling \mathcal{T} arising from a sequence $x = (x_n)$ in the space X and two intervals J_0 and J_1 . Note that the tiling dynamical system, which is denoted by $\mathbf{X} = (X_{\mathcal{T}}, \mathcal{D}, \nu, (T_t)_{(t \in \mathbb{R})})$, is minimal and uniquely ergodic since (X, T) is minimal and uniquely ergodic.

Theorem 5.1. *Let $\mathbf{X} = (X_{\mathcal{T}}, \mathcal{D}, \nu, (T_t)_{(t \in \mathbb{R})})$ be a tiling dynamical system arising from the subshift (X, T) and two intervals J_0 and J_1 . If $\frac{|J_0|}{|J_1|}$ is irrational, then the tiling dynamical system \mathbf{X} is rigid and weakly mixing.*

We will prove this for the case $|J_0| > |J_1|$. The proof for the other case ($|J_0| < |J_1|$) is analogous and will be omitted. In addition we assume that J_0 and J_1 have length 1 and α , where α is irrational and $0 < \alpha < 1$. We can do so because of the following fact: if $|\tilde{J}_0| = c|J_0|$ and $|\tilde{J}_1| = c|J_1|$ for some $c > 0$, then λ is an eigenvalue of a tiling system with two intervals J_0, J_1 if and only if $\frac{\lambda}{c}$ is an eigenvalue of a tiling system with two intervals \tilde{J}_0, \tilde{J}_1 .

Before proving Theorem 5.1, let us show the following lemma by an argument similar to that utilized in the proof of Lemma 1 in [JuPa].

Lemma 5.1. *T_α is an ergodic transformation on $(X_{\mathcal{T}}, \mathcal{D}, \nu)$.*

Proof. Let $h_m = l(B_m)$. For each m , denote $B_m = p_1^m \cdots p_{h_m}^m$ and form a new alphabet p_i^m ($1 \leq i \leq h_m$) as in the proof of Theorem 3.1. Then the countable set of patches

$$\begin{aligned} \{[p_i^m] \times [0, r) : r \in \mathbb{Q} \text{ with } 0 < r < |J_a| \text{ for } p_i^m = a \\ \text{or } 0 < r < |J_b| \text{ for } p_i^m = b, 1 \leq i \leq l(A_m), m \in \mathbb{N}\} \end{aligned}$$

generates \mathcal{D} .

Consider a cylinder set $P = [p_i^n] \times [0, r)$. Pick a tiling $\mathcal{S} \in X_{\mathcal{T}}$. We will show that T_α satisfies the ergodic theorem by showing that $T_\alpha^k(\mathcal{S})$ hits the cylinder set P with the right frequency.

Let $R_n := I_1 I_2 \cdots I_{h_n}$ be the patch corresponding to the block $B_n = x_1 x_2 \cdots x_{h_n}$ such that I_i is equivalent to J_{x_i} . For large s , consider a patch R_s . By Lemma 1.2 in [JuRo], for a fixed $n < s$, R_s can be written uniquely as a concatenation of strings of the form $R_n^{2^n}$ or $R_n^{2^{n+1}}$, separated by single J_1 s.

We estimate the number of $T_\alpha^k(\mathcal{S})$ hitting the cylinder set P by considering the translation $S_\alpha : x \rightarrow x + \alpha$ on the patch R_s . There exists an interval $L \subset R_n$ and a point $y \in R_s$ such that $T_\alpha^k(\mathcal{S})$ hitting the cylinder set P along the patch R_s is the same as $S_\alpha^k(x)$ hitting the interval L . Then the sequence of numbers where $T_\alpha^k(\mathcal{S})$ hits the cylinder set P in R_s converges as $s \rightarrow \infty$ since

- (1) the irrational rotation $x \rightarrow x + \alpha$ on the torus $[0, |R_n|)$ is uniquely ergodic,
- (2) $S_\alpha^k(x)$ hits J_1 only once whenever it meets the interval J_1 ,
- (3) J_1 appears in the patch R_s with uniform frequency, which tends to 0 as $s \rightarrow \infty$,

- (4) If x' and x'' have the same relative positions in the first R_n of the patches $R_n R_n$ and $R_n J_1 R_n$ respectively, then the relative positions of $S_\alpha^k(x')$ and $S_\alpha^{k+1}(x'')$ have the same relative positions when they occur in the second R_n of the $R_n R_n$ and $R_n J_1 R_n$ respectively, since $|J_1| = \alpha$.

Hence the frequency

$$\lim_{N \rightarrow \infty} \frac{1}{N} \sum_{n=1}^N 1_P(T_\alpha^{-k}(\mathcal{S})) = \lim_{N \rightarrow \infty} \frac{|\{1 \leq k \leq N : T_\alpha^k(\mathcal{S}) \in P\}|}{N}$$

exists. □

Proof of Theorem 5.1. Let $h_n = l(B_n)$ be the length of the block B_n . Note that

$$h_{n+1} = 2^{n+1}h_n + 1 \text{ and } \lim_{n \rightarrow \infty} h_n \mu([B_n]) = 1. \quad (4)$$

The second equation is obtained from Lemma 1.2 and Corollary 1.7 in [JuRo]: By Lemma

1.2 therein, $h_n \mu([B_n]) = \bigcup_{i=0}^{h_n-1} T^i([B_n])$ and by Corollary 1.7 of that paper we have

$$\lim_{n \rightarrow \infty} \mu(X \setminus \bigcup_{i=0}^{h_n-1} T^i([B_n])) = 0.$$

Let \mathcal{T}_n be the patch corresponding to the block B_n and let t_n be the diameter of \mathcal{T}_n . Let α_n and β_n be the numbers of 0s and 1s in B_n . Note that $\alpha_1 = 2, \beta_1 = 1, \alpha_{n+1} = 2^{n+1}\alpha_n$ and $\beta_{n+1} = 1 + 2^{n+1}\beta_n$. Then $(\frac{\beta_n}{\alpha_n})$ is a bounded increasing sequence, since $\frac{\beta_{n+1}}{\alpha_{n+1}} = \frac{1}{2^{n+1}\alpha_n} + \frac{\beta_n}{\alpha_n} \leq \sum_{k=1}^n \frac{1}{2^k} = 1$. Thus,

$$\lim_{n \rightarrow \infty} \frac{t_n}{h_n} = \lim_{n \rightarrow \infty} \frac{\alpha_n |J_0| + \beta_n |J_1|}{\alpha_n + \beta_n} = \lim_{n \rightarrow \infty} \frac{|J_0| + (\beta_n/\alpha_n) |J_1|}{1 + (\beta_n/\alpha_n)} = c, \quad (5)$$

for some $c > 0$.

Let us show first that \mathbf{X} is rigid. Let $C = [u] \times [0, r)$ for a finite word u and some $0 < r < |J_{u_0}|$. Since the measures of $[u]$ and C are given by the occurrence frequency, (3) in Section 3.2 holds, so

$$\nu(C) = \nu([u] \times [0, r)) = \mu([u]) \frac{r}{\mu([0])|J_0| + \mu([1])|J_1|}.$$

Given $y = (y_n) \in X$, let $\mathcal{T}_y := \{I_i : i \in \mathbb{Z}\}$ be the tiling of \mathbb{R} corresponding the sequence y . If $x \in [u] \cap T^{h_n}[u]$, then $T_t(\mathcal{T}_x) \in C \cap T_{t_n}C$ for $0 \leq t < r$, which implies that

$$\nu(C \cap T_{t_n}C) \geq \mu([u] \cap T^{h_n}[u]) \frac{r}{\mu([0])|J_0| + \mu([1])|J_1|}.$$

From Corollary 1.9 in [JuRo], (h_n) is a rigidity sequence for (X, T) , so $\lim_{n \rightarrow \infty} 1_{[u]} \circ T^{h_n} = 1_{[u]}$ for any finite word u in X . This implies that

$$\liminf \nu(C \cap T_{t_n}C) \geq \mu([u]) \frac{r}{\mu([0])|J_0| + \mu([1])|J_1|} = \nu(C),$$

so \mathbf{X} is rigid.

Now let us prove that $\mathbf{X} = (X_{\mathcal{T}}, \mathcal{D}, \nu, (T_t)_{(t \in \mathbb{R})})$ is weakly mixing. Let f be an eigenfunction of $(T_t)_{t \in \mathbb{R}}$ with $T_t f = e^{2\pi i \lambda t} f$ and $|f| = 1$. Since $\{B_n \times [a, b) : n \in \mathbb{N}, a, b \in \mathbb{Q}^+\}$ generates the Borel σ -algebra \mathcal{D} , given $\epsilon > 0$ we can find a positive integer n and a bounded measurable function f' such that f' is a finite linear combination of indicator functions of cylinder sets and $\int |f - f'| d\nu \leq \epsilon$. Note that f' is constant on $B_n \times \{s\}$ for any s .

Let $E_n = \{\mathcal{S} : \mathcal{S} \in B_n^{2^n} 1 B_n^{2^n} \times [0, 2^n t_n)\}$ and $F_n = \{\mathcal{S} : \mathcal{S} \in B_n^{2^n} B_n^{2^n} \times [0, 2^n t_n)\}$. Note that $T_{2^n t_n + \alpha} f'(\mathcal{S}) = f'(\mathcal{S})$ for $\mathcal{S} \in E_n$ and $T_{2^n t_n} f'(\mathcal{S}) = f'(\mathcal{S})$ for $\mathcal{S} \in F_n$. Let us show that $\liminf \nu(E_n) \geq \frac{1}{2}d$ and $\liminf \nu(F_n) \geq \frac{1}{2}d$, where $d = \frac{c}{\mu([0])|J_0| + \mu([1])|J_1|}$. Since $B_{n+1} = B_n^{2^n} 1 B_n^{2^n}$,

$$\nu(E_n) \geq \mu(B_{n+1}) \frac{2^n t_n}{\mu([0])|J_0| + \mu([1])|J_1|} = \frac{1}{2} \mu(B_{n+1}) h_{n+1} \frac{2^{n+1} h_n}{h_{n+1}} \frac{t_n/h_n}{\mu([0])|J_0| + \mu([1])|J_1|}.$$

By (4), we have $\lim_{n \rightarrow \infty} \mu(B_{n+1}) h_{n+1} = 1$ and $\lim_{n \rightarrow \infty} \frac{2^{n+1} h_n}{h_{n+1}} = 1$. By (5), we have $\lim_{n \rightarrow \infty} \frac{t_n}{h_n} = c$. Thus,

$$\liminf_{n \rightarrow \infty} \nu(E_n) \geq \frac{1}{2}d.$$

Note that $B_{n+1} B_{n+1} = B_n^{2^n} 1 B_n^{2^n} B_n^{2^n} 1 B_n^{2^n}$ contains $B_n^{2^n} B_n^{2^n}$ and the length of the patch corresponding to the block $B_n^{2^n}$ is $2^n t_n$, so

$$\nu(F_n) \geq \mu(B_{n+1} B_{n+1}) \frac{2^n t_n}{\mu([0])|J_0| + \mu([1])|J_1|}.$$

Also $B_{n+1} B_{n+1}$ occurs $2(2^{n+1} - 1)$ -times in B_{n+2} , so $\mu(B_{n+1} B_{n+1}) \geq \mu(B_{n+2}) 2(2^{n+1} - 1)$. Hence,

$$\begin{aligned} \nu(F_n) &\geq \mu(B_{n+1} B_{n+1}) \frac{2^n t_n}{\mu([0])|J_0| + \mu([1])|J_1|} \geq \mu(B_{n+2}) 2(2^{n+1} - 1) \frac{2^n t_n}{\mu([0])|J_0| + \mu([1])|J_1|} \\ &= \frac{1}{2} \mu(B_{n+2}) h_{n+2} \frac{(2^{n+2} - 2) 2^{n+1} h_n}{h_{n+2}} \frac{t_n/h_n}{\mu([0])|J_0| + \mu([1])|J_1|}. \end{aligned}$$

By (4), we have $\lim_{n \rightarrow \infty} \mu(B_{n+2}) h_{n+2} = 1$ and $\lim_{n \rightarrow \infty} \frac{(2^{n+2} - 2) 2^{n+1} h_n}{h_{n+2}} = \frac{(2^{n+2} - 2) h_{n+1}}{h_{n+2}} \frac{2^{n+1} h_n}{h_{n+1}} = 1$.

By (5), we have $\lim_{n \rightarrow \infty} \frac{t_n}{h_n} = c$. Hence,

$$\liminf_{n \rightarrow \infty} \nu(F_n) \geq \frac{1}{2}d.$$

Then,

$$\begin{aligned} |1 - e^{2\pi i \lambda (2^n t_n + \alpha)}| \nu(E_n) &= \int_{E_n} |f - T_{2^n t_n + \alpha} f| d\nu \\ &\leq \int_{X_{\mathcal{T}}} |f - f'| d\nu + \int_{E_n} |f' - T_{2^n t_n + \alpha} f'| d\nu + \int_{X_{\mathcal{T}}} |T_{2^n t_n + \alpha} f' - T_{2^n t_n + \alpha} f| d\nu \\ &\leq \int_{X_{\mathcal{T}}} |f - f'| d\nu + \int_{X_{\mathcal{T}}} |T_{2^n t_n + \alpha} f' - T_{2^n t_n + \alpha} f| d\nu \leq 2\epsilon, \end{aligned}$$

and

$$\begin{aligned}
|1 - e^{2\pi i \lambda 2^n t_n}| \nu(F_n) &= \int_{F_n} |f - T_{2^n t_n} f| d\nu \\
&\leq \int_{X_{\mathcal{T}}} |f - f'| d\nu + \int_{F_n} |f' - T_{2^n t_n} f'| d\nu + \int_{X_{\mathcal{T}}} |T_{2^n t_n} f' - T_{2^n t_n} f| d\nu \\
&\leq \int_{X_{\mathcal{T}}} |f - f'| d\nu + \int_{X_{\mathcal{T}}} |T_{2^n t_n} f' - T_{2^n t_n} f| d\nu \leq 2\epsilon.
\end{aligned}$$

Hence,

$$|e^{2\pi i \lambda \alpha} - 1| = |e^{2\pi i \lambda (2^n t_n + \alpha)} - e^{2\pi i \lambda 2^n t_n}| \leq |1 - e^{2\pi i \lambda (2^n t_n + \alpha)}| + |1 - e^{2\pi i \lambda 2^n t_n}| \leq \frac{2\epsilon}{\nu(E_n)} + \frac{2\epsilon}{\nu(F_n)}.$$

From $\liminf_{n \rightarrow \infty} \nu(E_n) \geq \frac{1}{2}d$ and $\liminf_{n \rightarrow \infty} \nu(F_n) \geq \frac{1}{2}d$,

$$|e^{2\pi i \lambda \alpha} - 1| \leq \frac{8}{d}\epsilon.$$

Since ϵ is arbitrarily small, $e^{2\pi i \lambda \alpha} = 1$. Then $T_{\alpha} f = f$. Ergodicity of T_{α} implies that f is constant. Hence the dynamical system is weakly mixing. \square

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